

A Novel Description of Linear Time–Invariant Networks via Structured Coprime Factorizations

Șerban Sabău, Cristian Oară, Sean Warnick and Ali Jadbabaie

Abstract

In this paper we study state–space realizations of Linear and Time–Invariant (LTI) systems. Motivated by biochemical reaction networks, Gonçalves and Warnick have recently introduced the notion of a *Dynamical Structure Functions* (DSF), a particular factorization of the system’s transfer function matrix that elucidates the interconnection structure in dependencies between manifest variables. We build onto this work by showing an intrinsic connection between a DSF and certain sparse left coprime factorizations. By establishing this link, we provide an interesting systems theoretic interpretation of sparsity patterns of coprime factors. In particular we show how the sparsity of these coprime factors allows for a given LTI system to be implemented as a network of LTI sub–systems. We examine possible applications in distributed control such as the design of a LTI controller that can be implemented over a network with a pre–specified topology.

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I. INTRODUCTION

Distributed and decentralized control of LTI systems has been a topic of intense research focus in control theory for more than 40 years. Pioneering work includes that of Radner [1], who revealed the sufficient conditions under which the minimal quadratic cost for a linear system can be achieved by a linear controller. Ho and Chu [2], laid the foundation of team theory by introducing a general class of distributed structures, dubbed *partially nested*, for which they showed the optimal LQG controller to be linear. More recently in [11], [12], [13], [14] important advances were made for the case where the decentralized nature of the problem is modeled as sparsity constraints on the input-output operator (the transfer function matrix) of the controller. These types of constraints are equivalent with computing the output feedback control law while having access to only partial measurements. Quite different from this scenario, in this work we are studying the meaning of sparsity constraints on the left coprime factors of the controller, which is not noticeable on its transfer function. In particular, we show how the sparsity of these coprime factors allows for the given LTI controller to be implemented over a LTI network with a pre-specified topology.

More recently, network reconstruction of biochemical reaction networks have motivated a careful investigation into the nature of systems and the many interpretations of *structure* or *sparsity structure* one may define [18]. In this work, a novel partial structure representation for Linear Time-Invariant (LTI) systems, called the *Dynamical Structure Function* (DSF) was introduced. The DSF was shown to be a factorization of a system's transfer function that represented the *open-loop causal dependencies among manifest variables*, an interpretation of system structure dubbed the *Signal Structure*.

A. An Introductory Example [18]

One important characteristic of the DSF is its ability to represent the impact that observed variables have on each other. This can often effectively describe the interconnection structure between component subsystems within a given system. Consider for example the 3-hop ring (also called “delta”) network in Figure 1, where all the $Q(s)$ and $P(s)$ blocks represent transfer functions of continuous-time LTI systems. We denote with $L(s)$ the transfer function from the input signals $U(s)$ to the outputs $Y(s)$. By directly inspecting the signal flow graph in Figure 1 we can write the algebraic equations:

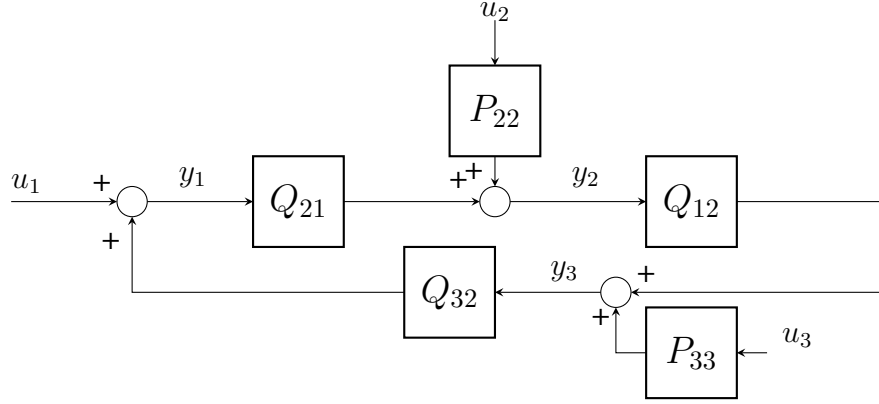


Fig. 1. A 3-Hop Ring Network

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix} = \begin{bmatrix} O & O & Q_{13}(s) \\ Q_{21}(s) & O & O \\ O & Q_{32}(s) & O \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix} + \begin{bmatrix} I & O & O \\ O & P_{22}(s) & O \\ O & O & P_{33}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix} \quad (1)$$

We make the additional notation

$$Q(s) \stackrel{def}{=} \begin{bmatrix} O & O & Q_{13}(s) \\ Q_{21}(s) & O & O \\ O & Q_{32}(s) & O \end{bmatrix} \quad \text{and} \quad P(s) \stackrel{def}{=} \begin{bmatrix} I & O & O \\ O & P_{22}(s) & O \\ O & O & P_{33}(s) \end{bmatrix} \quad (2)$$

and we define ad-hoc the $(Q(s), P(s))$ pair to be the Dynamical Structure Function associated with the $L(s)$ LTI system. (The rigorous definition of DFS will be introduced in Section II following the original mathematical derivation from [18].) An interesting observation, which is the main thesis of this work, is that the structure of the subsystems interconnections in Figure 1 is no longer recognizable from the input-output relation described by the transfer function of the aggregate system $L(s) = (I_3 - Q(s))^{-1}P(s)$ since the transfer function $L(s)$ does not have any sparsity pattern and in general does not have any other particularities. The structure however, remains visible and it is captured in the quite particular sparsity patterns of $Q(s)$ and $P(s)$, respectively. This key property makes the DSF susceptible of becoming a perfectly suited theoretical concept to model any LTI network.

We want to illustrate further how the DSF determines via equation (16) the topology of the LTI network that can describe the given LTI system $L(\lambda)$. If we consider $Q_{13}(s)$ identically zero

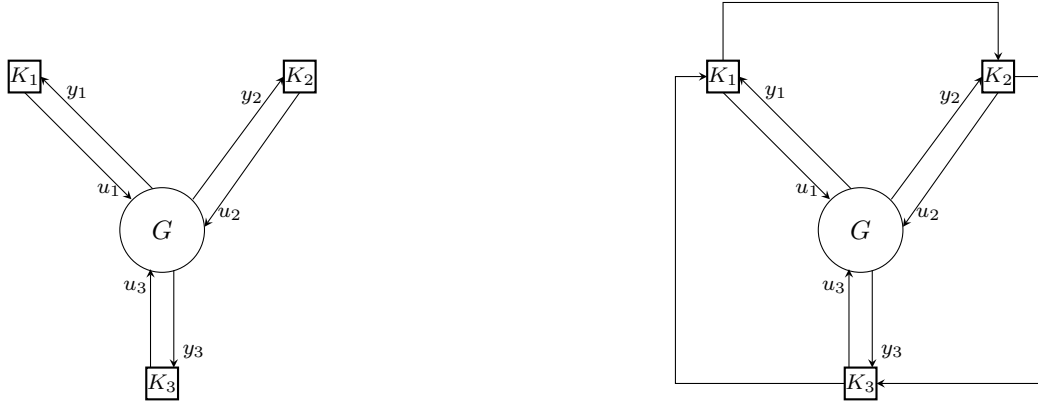


Fig. 2. The Plant G and the Decentralized (Diagonal) Controller (left) versus a “Ring” Networked Controller (right)

in (1) which would mean “breaking” the ring network from Figure 1 then it becomes a cascade connection and $L(\lambda)$ can be implemented as a “line” network. A “line” network controller could be interesting for example motion control of vehicles moving in a platoon formation.

Note that, in general, the impact of observed variables on each other, represented by the DSF, and the interconnection between subsystems, can be quite different structures. This is because the states internal to one subsystem are always distinct from another, while the states internal to component systems in the DSF may be shared with other components. Nevertheless, the point in this example, that the DSF, as a factorization of a system’s transfer function, captures an important notion of structure, is always true. Details about the distinctions between subsystem structure and the signal structure described by the DSF can be found in [19].

B. Motivation and Scope of Work

In this paper we look at Dynamical Structure Functions from a control systems perspective. A long standing problem in control of LTI systems was synthesis of decentralized stabilizing controllers ([4]) which means imposing on the controller’s transfer function matrix $K(s)$ to have a diagonal sparsity pattern. Quite different to the decentralized paradigm, the ultimate goal of our research would be a systematic method of designing controllers that can be implemented as a LTI network with a pre-specified topology. This is equivalent with computing a stabilizing controller $K(s)$ whose DSF $(Q(s), P(s))$ satisfies certain sparsity constraints [20]. So, instead of imposing sparsity constraints on the transfer function of the controller as it is the case in decentralized

control, we are interested in imposing the sparsity constraints on the controller's DSF. This would eventually lead to the possibility of designing controllers that can be implemented as a LTI network, see for example Figure 2.

C. Contribution

The contribution of this paper is the establishment of the intrinsic connections between the DSFs and the left coprime factorizations of a given transfer function and to give a systems theoretic meaning to sparsity patterns of coprime factors using DSFs. The importance of this is twofold. First, this is the most common scenario in control engineering practice (*e.g.* manufacturing, chemical plants) that the given plant is made out of many interconnected sub-systems. The *structure* of this interconnection is captured by a DFS description of the plant which in turn might translate to left coprime factorization of the plant that features certain sparsity patterns on its factors. This sparsity might be used for the synthesis of a controller to be implemented over a LTI network. Conversely, in many applications it is desired that the stabilizing controller be implemented in a distributed manner, for instance as a LTI network with a pre-specified topology. This is equivalent to imposing certain sparsity constraints on the left coprime factorization of the controller (via the celebrated Youla parameterization). In order to fully exploit the power of the DSFs approach to tackle these types of problems, we find it useful to underline its links with the classical notions and results in control theory of LTI systems. We provide here a comprehensive exposition of the elemental connections between the Dynamical Structure Functions and the Coprime Factorizations of a given Linear Time-Invariant (LTI) system, thus opening the way between exploiting the structure of the plant via the DSF and employing the celebrated Youla parameterization for feedback output stabilization.

D. Outline of the Paper

In the second Section of the paper we give a brief outline of the theoretical concept of Dynamical Structure Functions as originally introduced in [18]. In the third Section, we show that while the DSF representation of a given LTI system $L(s)$ is in general never coprime, a closely related representation dubbed a viable (W, V) pair associated with $L(s)$ is always coprime. We also provide the class of all viable (W, V) pairs associated with a given $L(s)$. The fourth Section contains the main results of the paper and it makes a complete explanation of

the natural connections between the DSFs and the viable (W, V) pairs associated with a given $L(s)$ and its left coprime factorizations. The last Section contains the conclusions and future research directions. In the Appendix A we have provided a short primer on realization theory for improper TFMs which is indispensable for the proofs of the main results. The proofs of the main results have been placed in Appendix B.

II. DYNAMICAL STRUCTURE FUNCTIONS

The main object of study here is a LTI system, which in the continuous-time case are described by the state equations

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(t_o) = x_o \quad (3a)$$

$$y(t) = Cx(t) + Du(t) \quad (3b)$$

where A, B, C, D are $n \times n, n \times m, p \times n, p \times m$ real matrices, respectively while n is also called *the order* of the realization. Given any n -dimensional state-space representation (3a), (3b) of a LTI system (A, B, C, D) , its input-output representation is given by the *Transfer Function Matrix* (TFM) which is the $p \times m$ matrix with real, rational functions entries denoted with

$$L(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \stackrel{\text{def}}{=} D + C(\lambda I_n - A)^{-1}B, \quad (4)$$

Remark II.1. *Our results apply on both continuous or discrete time LTI systems, hence we assimilate the undeterminate λ with the complex variables s or z appearing in the Laplace or Z-transform, respectively, depending on the type of the system.*

For elementary notions in linear systems theory, such as state equivalence, controllability, observability, detectability, we refer to [8], or any other standard text book on LTI systems.

By $\mathbb{R}^{p \times m}$ we denote the set of $p \times m$ real matrices and by $\mathbb{R}(\lambda)^{p \times m}$ we denote $p \times m$ transfer function matrices (matrices having entries real-rational functions).

This section contains a discussion based on reference [18] on the definition of the Dynamical Structure Functions associated with a LTI system. We start with the given system $L(\lambda)$ described

by the following state equations, of order n :

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t); \quad \tilde{x}(t_o) = \tilde{x}_o \quad (5a)$$

$$y(t) = \tilde{C}\tilde{x}(t) \quad (5b)$$

Assumption II.2. (Regularity) *We make the assumption that the \tilde{C} matrix from (5b) has full row rank (it is surjective).*

We choose any matrix \bar{C} such that $T \stackrel{def}{=} \begin{bmatrix} \tilde{C} \\ \bar{C} \end{bmatrix}$ is nonsingular (note that such \bar{C} always exists because \tilde{C} has full row rank) and apply a state-equivalence transformation

$$x(t) = T\tilde{x}(t), \quad A = T\tilde{A}T^{-1}, \quad B = T\tilde{B}, \quad C = \tilde{C}T^{-1}. \quad (6)$$

on (5a),(5b) in order to get

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = T\tilde{x}(t) \quad (7a)$$

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t); \quad \begin{bmatrix} y(t_o) \\ z(t_o) \end{bmatrix} = \begin{bmatrix} y_o \\ z_o \end{bmatrix} \quad (7b)$$

$$y(t) = \begin{bmatrix} I_p & O \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \quad (7c)$$

Assumption II.3. (Observability) *We can assume without any loss of generality that the pair (\tilde{C}, \tilde{A}) from (5a), (5b) or equivalently the pair (A_{12}, A_{22}) from (7b) are observable.*

Remark II.4. *The argument that the observability assumption does not imply any loss of generality, is connected with the Leuenberger reduced order observer.*

Looking at the Laplace or Z-transform of the equation in (7b), we get

$$\begin{bmatrix} \lambda I_p - A_{11} & -A_{12} \\ -A_{21} & \lambda I_{n-p} - A_{22} \end{bmatrix} \begin{bmatrix} Y(\lambda) \\ Z(\lambda) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(\lambda) \quad (8)$$

By multiplying (8) from the left with the following factor $\Omega(\lambda)$

$$\Omega(\lambda) = \begin{bmatrix} I_p & A_{12}(\lambda I_{n-p} - A_{22})^{-1} \\ O & I_{n-p} \end{bmatrix} \quad (9)$$

(note that $\Omega(\lambda)$ is always invertible as a TFM) we get

$$\begin{bmatrix} \left((\lambda I_p - A_{11}) - A_{12}(\lambda I_{n-p} - A_{22})^{-1} A_{21} \right) & O \\ * & * \end{bmatrix} \begin{bmatrix} Y(\lambda) \\ Z(\lambda) \end{bmatrix} = \Omega(\lambda) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(\lambda) \quad (10)$$

where the $*$ denote entries whose exact expression is not needed now. Immediate calculations yield that the first block-row in (10) is equivalent with

$$\lambda Y(\lambda) = \left(A_{11} + A_{12}(\lambda I_{n-p} - A_{22})^{-1} A_{21} \right) Y(\lambda) + \left(B_1 + A_{12}(\lambda I_{n-p} - A_{22})^{-1} B_2 \right) U(\lambda) \quad (11)$$

and by making the notation

$$W(\lambda) \stackrel{def}{=} -A_{11} - A_{12}(\lambda I_{n-p} - A_{22})^{-1} A_{21} \quad (12a)$$

$$V(\lambda) \stackrel{def}{=} B_1 + A_{12}(\lambda I_{n-p} - A_{22})^{-1} B_2 \quad (12b)$$

we finally get the following equation which describes the relationship between manifest variables

$$\lambda Y(\lambda) = W(\lambda)Y(\lambda) + V(\lambda)U(\lambda). \quad (13)$$

Remark II.5. *Note that if $V(\lambda)$ is identically zero, while $W(\lambda)$ is a constant matrix having the sparsity of a graph's Laplacian, then (13) becomes the free evolution equation $\lambda Y(\lambda) = WY(\lambda)$. These types of equations have been extensively studied in cooperative control [15] to describe the dynamics of a large group of autonomous agents. Equation (13) can be looked at as a generalization of that model and will be studied here in a different context.*

Since $L(\lambda)$ is the input-output operator from $U(\lambda)$ to $Y(\lambda)$, we can write equivalently that $L(\lambda) = (\lambda I_p - W(\lambda))^{-1} V(\lambda)$, which is exactly the (W, V) representation from [18, (3)/ pp.1671]. (Note that since $W(\lambda)$ is always proper it follows that $(\lambda I_p - W(\lambda))$ is always invertible as a TFM.) Next, let $D(\lambda)$ denote the TFM obtained by taking the diagonal entries of $W(\lambda)$, that

is $D(\lambda) \stackrel{\text{def}}{=} \text{diag}\{W_{11}(\lambda), W_{22}(\lambda) \dots W_{pp}(\lambda)\}$. Then we can write $L(\lambda) = \left[\left(\lambda I_p - D(\lambda) \right) - \left(W(\lambda) - D(\lambda) \right) \right]^{-1} V(\lambda)$, or equivalently (note that $(W - D)$ has zeros on the diagonal entries)

$$L(\lambda) = \left[I - \left(\lambda I_p - D(\lambda) \right)^{-1} \left(W(\lambda) - D(\lambda) \right) \right]^{-1} \left(\lambda I_p - D(\lambda) \right)^{-1} V(\lambda) \quad (14)$$

and after introducing the notation

$$Q(\lambda) \stackrel{\text{def}}{=} \left(\lambda I_p - D(\lambda) \right)^{-1} \left(W(\lambda) - D(\lambda) \right) \quad (15a)$$

$$P(\lambda) \stackrel{\text{def}}{=} \left(\lambda I_p - D(\lambda) \right)^{-1} V(\lambda) \quad (15b)$$

we get that $L(\lambda) = \left(I_p - Q(\lambda) \right)^{-1} P(\lambda)$ or equivalently that

$$Y(\lambda) = Q(\lambda)Y(\lambda) + P(\lambda)U(\lambda) \quad (16)$$

Remark II.6. *The splitting and the “extraction” of the diagonal in (15a) are made in order to make the $Q(\lambda)$ have the sparsity (and the meaning) of the adjacency matrix of the graph describing the causal relationships between the manifest variables $Y(\lambda)$. Consequently, $Q(\lambda)$ will always have zero entries on its diagonal.*

Definition II.7. *[18, Definition 1] Given the state–space realization (7b),(7c) of $L(\lambda)$ the Dynamical Structure Function of the system is defined to be the pair $(Q(\lambda), P(\lambda))$, where $Q(\lambda), P(\lambda)$ are given by (15a) and (15b) respectively.*

III. DYNAMICAL STRUCTURE FUNCTIONS REVISITED

One scope of this paper and also one of its contributions is to emphasize the idea that for a given TFM $L(\lambda)$ there exist more than one pair $(Q(\lambda), P(\lambda))$ than the one in (15a),(15b) (originally introduced in [18]) and which satisfy (16). In fact there exists a whole class of pairs $(Q(\lambda), P(\lambda))$ that do satisfy (16) and for which $Q(\lambda)$ has all its block–diagonal entries equal to zero. In order to illustrate this we need to slightly reformulate the original Definition II.7 of Dynamical Structure Functions associated with a $L(\lambda)$ as follows:

Definition III.1. Given a TFM $L(\lambda)$, we define a Dynamical Structure Function representation of $L(\lambda)$ to be any two TFMs $Q(\lambda) \in \mathbb{R}^{p \times p}(\lambda)$ and $P(\lambda) \in \mathbb{R}^{p \times m}(\lambda)$ with $Q(\lambda)$ having zero entries on its diagonal, such that $L(\lambda) = (I_p - Q(\lambda))^{-1}P(\lambda)$ or equivalently

$$Y(\lambda) = Q(\lambda)Y(\lambda) + P(\lambda)U(\lambda) \quad (17)$$

The following definition will also be needed in the sequel.

Definition III.2. Given the TFM $L(\lambda)$, we call a viable $(W(\lambda), V(\lambda))$ pair associated with $L(\lambda)$, any two TFMs $W(\lambda) \in \mathbb{R}^{p \times p}(\lambda)$ and $V(\lambda) \in \mathbb{R}^{p \times m}(\lambda)$, with $W(\lambda)$ having McMillan degree at most $(n - p)$ and such that

$$L(\lambda) = \left(\lambda I_p - W(\lambda) \right)^{-1} V(\lambda). \quad (18)$$

Proposition III.3. Given a TFM $L(\lambda)$ then for any given viable $(W(\lambda), V(\lambda))$ pair associated with $L(\lambda)$, there exists a unique DSF representation $(Q(\lambda), P(\lambda))$ of $L(\lambda)$ given by (15a) and (15b), where $D(\lambda) \stackrel{\text{def}}{=} \text{diag}\{W_{11}(\lambda), W_{22}(\lambda) \dots W_{pp}(\lambda)\}$ is uniquely determined by $W(\lambda)$.

Proof: The proof follows immediately from the very definitions (15a), (15b). ■

Remark III.4. It is important to remark here that any viable $(W(\lambda), V(\lambda))$ pair has the same sparsity pattern with its subsequent DSF representation $(Q(\lambda), P(\lambda))$. For example $W(\lambda)$ is lower triangular if and only if $Q(\lambda)$ is lower triangular. Similarly, for instance $V(\lambda)$ is tridiagonal if and only if $P(\lambda)$ is tridiagonal.

Remark III.5. Using Proposition III.3 we can conclude that in order to find all DSFs (according to Definition III.1) associated with a given $L(\lambda)$, it is sufficient to study the set of all viable $(W(\lambda), V(\lambda))$ pairs associated with $L(\lambda)$. The following theorem gives closed-formulas for the parameterization of the class of all viable $(W(\lambda), V(\lambda))$ pairs associated with a given TFM.

Theorem III.6. Given a TFM $L(\lambda)$ having a state-space realization (5a),(5b), we compute any equivalent realization (7b),(7c). The class of all viable $(W(\lambda), V(\lambda))$ pairs associated with $L(\lambda)$ is then given by

$$-W(\lambda) = \left[\begin{array}{c|c} (A_{22} + KA_{12}) - \lambda I_{n-p} & A_{22}K + KA_{12}K - KA_{11} - A_{21} \\ \hline A_{12} & -A_{11} + A_{12}K \end{array} \right] \quad (19)$$

$$V(\lambda) = \left[\begin{array}{c|c} (A_{22} + KA_{12}) - \lambda I_{n-p} & KB_1 + B_2 \\ \hline A_{12} & B_1 \end{array} \right] \quad (20)$$

where the K is any matrix in $\mathbb{R}^{(n-p) \times p}$ and $A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2$ are as in (7b),(7c).

Proof: See Appendix B. ■

Remark III.7. We remark here the poles of both $W(\lambda)$ and $V(\lambda)$ can be allocated at will in the complex plane, by a suitable choice of the matrix K and the assumed observability of the pair (A_{12}, A_{22}) (Assumption II.3).

IV. MAIN RESULTS

The ultimate goal of this line of research would be computing controllers whose DSF has a certain structure. This would allow us for instance to compute controllers that can be implemented as a “ring” network (see Figure 1) or as a “line” network which is important for motion control of vehicles moving in a platoon formation. However, classical results in LTI systems control theory, such as the celebrated Youla parameterization (or its equivalent formulations) render the expression of the stabilizing controller as a stable coprime factorization of its transfer function. As a first step towards employing Youla-like methods for the synthesis of controllers featuring structured DSF, we need to understand the connections between the stable left coprime factorizations (of a given stabilizing controller) and its DSF representation. We address this problem in this section.

A. A Result on Coprimeness

In this subsection we prove that (by chance rather than by design) for any viable $(W(\lambda), V(\lambda))$ pair associated with a given $L(\lambda)$ (with $W(\lambda)$ and $V(\lambda)$ as in Theorem III.6) it follows that $(\lambda I_p - W(\lambda), V(\lambda))$ is a *left coprime factorization* of $L(\lambda)$. An equivalent condition for $(\lambda I_p - W(\lambda), V(\lambda))$ to be left coprime is for the compound transfer function matrix

$$\left[\begin{array}{cc} (\lambda I_p - W(\lambda)) & V(\lambda) \end{array} \right] \quad (21)$$

to have no (finite or infinite) Smith zeros (see [3], [9], [10] for equivalent characterizations of left coprimeness). Coprimeness is especially important for output feedback stabilization, since classical results such as the celebrated Youla parameterization, require a coprime factorization of the plant while also rendering coprime factors of the stabilizing controllers.

Assumption IV.1. (Controllability) *From this point onward we assume that the realization (5a),(5b) of $L(\lambda)$ is controllable.*

Theorem IV.2. *Given a TFM $L(\lambda)$, then for any viable $(W(\lambda), V(\lambda))$ pair associated with a given $L(\lambda)$ (with $W(\lambda)$ and $V(\lambda)$ as in Theorem III.6) it follows that $(\lambda I_p - W(\lambda), V(\lambda))$ is a left coprime factorization of $L(\lambda)$.*

Proof: See Appendix B. ■

Remark IV.3. *We remark here that while any viable $(W(\lambda), V(\lambda))$ pair associated with a given $T(\lambda)$ makes out for a left coprime factorization $L(\lambda) = (\lambda I_p - W(\lambda))^{-1}V(\lambda)$, the DSF $L(\lambda) = (I_p - Q(\lambda))^{-1}P(\lambda)$ are in general never coprime (unless the plant is stable or diagonal). That is due to the fact that in general not all the unstable zeros of $(\lambda I_p - D(\lambda))$ cancel out when forming the products in (15a), (15b) and the same unstable zeros will result in poles/zeros cancelations when forming the product $L(\lambda) = (I_p - Q(\lambda))^{-1}P(\lambda)$.*

B. Getting from DSFs to Stable Left Coprime Factorizations

In this subsection we show that for any viable pair $(W(\lambda), V(\lambda))$ with both $W(\lambda)$ and $V(\lambda)$, respectively being stable, there exists a class of stable left coprime factorizations. Furthermore, there exists a class of stable left coprime factorizations that preserve the sparsity pattern of the original viable pair $(W(\lambda), V(\lambda))$.

Note that for any viable pair $(W(\lambda), V(\lambda))$ is an *improper* rational function and it has exactly p poles at infinity of multiplicity one, hence the $(\lambda I_p - W(\lambda))$ factor (the denominator of the factorization) is inherently *unstable* (in either continuous or discrete-time domains). We remind the reader that any the poles of both $W(\lambda)$ and $V(\lambda)$ can be allocated at will in the stability domain (Remark III.7). In this subsection, we show how to get from viable pair $(W(\lambda), V(\lambda))$ of $L(\lambda)$ in which both factors $W(\lambda)$ and $V(\lambda)$ are stable, to a stable left coprime factorization $L(\lambda) = M^{-1}(\lambda)N(\lambda)$. We achieve this without altering any of the stable poles of

$W(\lambda)$ and $V(\lambda)$ (which are the modes of $(A_{22} + KA_{12})$ in (19), (20)) and while at the same time keeping the McMillan degree to the minimum. The problem is to displace the p poles at infinity (of multiplicity one) from the $(\lambda I_p - W(\lambda))$ factor. To this end we will use the *Basic Pole Displacement Result* from [10, Theorem 3.1] that shows that this can be achieved by premultiplication with an adequately chosen invertible factor $\Theta(\lambda)$ such that when forming the product $\Theta(\lambda)(\lambda I_p - W(\lambda))$ all the p poles at infinity of the factor $(\lambda I_p - W(\lambda))$ cancel out. Here follows the precise statement:

Lemma IV.4. *Given a viable pair $(\lambda I_p - W(\lambda), V(\lambda))$ of $L(\lambda)$ then for any*

$$\Theta(\lambda) \stackrel{\text{def}}{=} \left[\begin{array}{c|c} A_x - \lambda I_p & T_4 \\ \hline T_5 & O \end{array} \right] \quad (22)$$

with A_x, T_4, T_5 arbitrarily chosen such that A_x has only stable eigenvalues and both T_4, T_5 are invertible, it follows that

$$\left[\begin{array}{cc} M(\lambda) & N(\lambda) \end{array} \right] \stackrel{\text{def}}{=} \Theta(\lambda) \left[\begin{array}{cc} (\lambda I_p - W(\lambda)) & V(\lambda) \end{array} \right] \quad (23)$$

is a stable left coprime factorization $L(\lambda) = M^{-1}(\lambda)N(\lambda)$. Furthermore,

$$\left[\begin{array}{cc} M(\lambda) & N(\lambda) \end{array} \right] = \left[\begin{array}{cc|cc} A_x - \lambda I_{n-p} & T_4 A_{12} & (A_x T_4 - T_4 A_{11} + T_4 A_{12} K) & T_4 B_1 \\ O & A_{22} + K A_{12} - \lambda I_p & (A_{22} K + K A_{12} K - K A_{11} - A_{21}) & K B_1 + B_2 \\ \hline T_4^{-1} & O & I & O \end{array} \right] \quad (24)$$

hence all the modes in $(A_{22} + K A_{12})$ (which are the original stable poles of $W(\lambda)$ and $V(\lambda)$) are preserved in the $M(\lambda)$ and $N(\lambda)$ factors.

Proof: See Appendix B. ■

Remark IV.5. *We remark that for any diagonal A_x having only stable eigenvalues $\Theta(\lambda) = (\lambda I_p - A_x)^{-1}$ yields a stable left coprime factorization of $L(\lambda)$ that preserves the sparsity structure of the initial viable $(\lambda I_p - W(\lambda), V(\lambda))$ pair.*

C. Connections with the Nett & Jacobson Formulas [16]

In this subsection, we are interested in connecting the expression from (24) for the pair $(M(\lambda), N(\lambda))$ to the classical result of state-space derivation of left coprime factorizations of a given plant originally presented in [16] (and generalized in [17]).

Proposition IV.6. [16], [17] Let $L(\lambda)$ be an arbitrary $m \times p$ TFM and Ω a domain in \mathbb{C} . The class of all left coprime factorizations of $L(\lambda)$ over Ω , $T(\lambda) = M^{-1}(\lambda)N(\lambda)$, is given by

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = U^{-1} \left[\begin{array}{c|cc} (A - FC) - \lambda I & -F & B \\ \hline C & I & O \end{array} \right], \quad (25)$$

where A, B, C, F and U are real matrices accordingly dimensioned such that

- i) U is any $p \times p$ invertible matrix,
- ii) F is any feedback matrix that allocates the observable modes of the (C, A) pair to Ω ,
- iii) $L(\lambda) = \left[\begin{array}{c|c} A - \lambda I & B \\ \hline C & O \end{array} \right]$ is a stabilizable realization.

Due to Assumption IV.1, we have to replace the stabilizability from point **iii)** with a controllability assumption. We start off with $L(\lambda)$ given by the equations (5a),(5b)

$$L(\lambda) = \left[\begin{array}{cc|c} A_{11} - \lambda I_p & A_{12} & B_1 \\ A_{21} & A_{22} - \lambda I_{n-p} & B_2 \\ \hline I & O & O \end{array} \right] \quad (26)$$

and we want to retrieve (24) by using the parameterization in Proposition IV.6. First apply a state-equivalence $T = \begin{bmatrix} T_4 & O \\ K & I \end{bmatrix}$ in order to get

$$L(\lambda) = \left[\begin{array}{cc|c} T_4(A_{11} - A_{12}K)T_4^{-1} - \lambda I_p & T_4 A_{12} & T_4 B_1 \\ (KA_{11} + A_{21} - KA_{12}K - A_{22}K)T_4^{-1} & KA_{12} + A_{22} - \lambda I_{n-p} & KB_1 + B_2 \\ \hline T_4^{-1} & O & O \end{array} \right] \quad (27)$$

Next, we only need to identify the F feedback matrix from point **ii)** of Proposition IV.6, which in this case is proven to be given by

$$F = \begin{bmatrix} (T_4^{-1}A_x T_4) - A_{11} + A_{12}K \\ -K(T_4^{-1}A_x T_4) + A_{22}K - A_{21} \end{bmatrix} \quad (28)$$

To check, simply plug (28) in (25) for the realization (??) of $L(\lambda)$.

D. Getting from the Stable Left Coprime Factorization to the DSFs

In this subsection we show that for *almost* every stable left coprime factorization of a given LTI system, there is an associated a *unique* viable $(W(\lambda), V(\lambda))$ pair and consequently (via Remark III.3) a unique DSF representation $(Q(\lambda), P(\lambda))$. The key role in establishing this one to one correspondence is played by a non-symmetric Riccati equation, whose solution existence is a generic property. This result is meaningful, since for controller synthesis while we are interested in the DSF of the controller, in general we only have access to a stable left coprime of the controller.

We start with a given stable left coprime factorization (25) for $L(\lambda)$ having an order n realization

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = U^{-1} \left[\begin{array}{c|cc} (A - FC) - \lambda I & -F & B \\ \hline C & I & O \end{array} \right] \quad (29)$$

to which we apply a type (6) state-equivalence transformation with $T \in \mathbb{R}^{n \times n}$ such that $CT^{-1} = \begin{bmatrix} I_p & O \end{bmatrix}$. Note that such a T always exists because of Assumption II.2. It follows that (29) takes the form

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = \left[\begin{array}{cc|cc} A_{11} + F_1 - \lambda I_p & A_{12} & F_1 & B_1 \\ A_{21} + F_2 & A_{22} - \lambda I_{n-p} & F_2 & B_2 \\ \hline I_p & O & I_p & O \end{array} \right] \quad (30)$$

and denote

$$A^+ \stackrel{\text{def}}{=} \begin{bmatrix} A_{11} + F_1 & -A_{12} \\ -(A_{21} + F_2) & A_{22} \end{bmatrix} \quad (31)$$

The solution to the following nonsymmetric algebraic Riccati matrix equation is paramount to the main result of this subsection, since it underlines the one to one correspondence between (30) and its *unique* associated viable $(W(\lambda), V(\lambda))$ pair.

Proposition IV.7. *The nonsymmetric algebraic Riccati matrix equation*

$$K(A_{11} + F_1) - A_{22}K - KA_{12}K + (A_{21} + F_2) = O \quad (32)$$

has a stabilizing solution K (i.e. $(A_{11} + F_1 - A_{12}K)$ is stable) if and only if the A^+ matrix from (31) has a stable invariant subspace of dimension p with basis matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad (33)$$

having V_1 invertible (i.e. disconjugate). In this case $K = V_1^{-1}V_2$ and it is the unique solution of (32).

Proof: It follows from [22]. ■

Remark IV.8. Since in our case A^+ is stable, all its invariant subspaces are actually stable (including the whole space). Therefore, the Riccati equation has a stabilizing solution if and only if the matrix A^+ has an invariant subspace of dimension p which is disconjugate. Hence, if for example A^+ has only simple eigenvalues, the Riccati equation always has a solution (we can always select p eigenvectors (from the n eigenvectors) to form a disconjugate invariant subspace). In this case, all we have to do is to order the eigenvalues in a Schur form such that the corresponding invariant subspace has V_1 invertible. Although this is a generic property, when having Jordan blocks of dimension greater than one it might happen that the matrix A^+ has no disconjugate invariant subspace of appropriate dimension p , and therefore the Riccati equation has no solution (stable or otherwise).

Theorem IV.9. Given any stable left coprime factorization $L(\lambda) = M^{-1}(\lambda)N(\lambda)$ and its state-space realization (30), let K be the solution of the nonsymmetric algebraic Riccati equation (32) and denote $A_x \stackrel{\text{def}}{=} (F_1 + A_{11} - A_{12}K)$. Then, a state-space realization for $\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix}$ is given by

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = \left[\begin{array}{cc|cc} A_x - \lambda I_{n-p} & A_{12} & (A_x - A_{11} + A_{12}K) & B_1 \\ O & A_{22} + KA_{12} - \lambda I_p & (A_{22}K + KA_{12}K - KA_{11} - A_{21}) & KB_1 + B_2 \\ \hline I & O & I & O \end{array} \right]. \quad (34)$$

Furthermore, from (34) we can recover the exact expression of the subsequent viable $(W(\lambda), V(\lambda))$ pair associated with $L(\lambda)$, where $W(\lambda)$ and $V(\lambda)$ are given by (19) and (20), respectively.

Proof: For the proof, simply plug

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} A_x - A_{11} + A_{12}K \\ -KA_x + A_{22}K - A_{21} \end{bmatrix} \quad (35)$$

into the expression of (30) in order to obtain (34). The rest of the proof follows from Lemma IV.4, by taking T_4 to be equal with the identity matrix I_p . ■

Remark IV.10. *We remark here that in general there is no correlation between the sparsity pattern of the stable left coprime (30) we start with and its associated viable $(W(\lambda), V(\lambda))$ pair produced in Theorem IV.9. That is to say that the converse of the observation made in Remark IV.5 is not valid. This poses additional problems for controller synthesis, since it might happen to encounter stable left coprime factorizations that have no particular sparsity pattern (are dense TFMs) while their associated viable $(W(\lambda), V(\lambda))$ pair are sparse. This is due to the fact that in general, the A_x matrix in Theorem IV.9 can be a dense matrix. One way to circumvent this problem would be to use a carefully adapted version of Youla's parameterization in which the stable left coprime factorization to be replaced with a DSF description where both with $(W(\lambda), V(\lambda))$ factors are stable. This is the topic of our future investigation.*

V. CONCLUSIONS

In this paper we have presented an exhaustive discussion on the intrinsic connections between the DSFs associated with a given transfer function and its left coprime factorizations. We have showed that rather than dealing directly with the DSF representation it is more beneficial to work on the so-called viable $(W(\lambda), V(\lambda))$ pairs associated with a given system. This theoretical results ultimately aim at a method of designing LTI controllers that can be implemented over a network with a pre-specified topology. We currently have sufficient conditions for the existence of such controllers but we miss the necessary conditions. While in general these conditions might be very hard to find, we expect to find such conditions for plants featuring special DSF structures.

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APPENDIX A

Definition V.1. A TFM $L(\lambda)$ is called *improper* if for at least one of its entries (which are real-rational functions), it holds that the degree of the numerator is strictly larger than the degree of the denominator.

Proposition V.2. ([5], [7]) Any improper (even polynomial) $p \times m$ rational matrix $L(\lambda)$ with coefficients in \mathbb{R} has a descriptor realization of the form

$$L(\lambda) = D + C(\lambda E - A)^{-1}B =: \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array} \right], \quad (36)$$

where $A, E \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and the so called pole pencil $A - \lambda E$ is regular, i.e., it is square and $\det(A - \lambda E) \not\equiv 0$. The dimension n of the square matrices A and E is called the order of the realization (36).

Definition V.3. The descriptor realization (36) of $L(\lambda)$ is called *minimal* if its order is as small as possible among all realizations of this kind.

Definition V.4. The McMillan degree of $L(\lambda)$ – denoted $\delta(L)$ – is the sum of the orders of all the poles of $L(\lambda)$ (finite and infinite).

Remark V.5. The principal inconvenience of realizations of the form (36) is that their minimal possible order is greater than the McMillan degree of $L(\lambda)$, unless $L(\lambda)$ is proper, and this brings important technical difficulties in factorization problems in which the McMillan degree plays a paramount role. A remedy to this is to use a generalization of (36) in which either the “ B ” or the “ C ” matrix is replaced by a matrix pencil, as stated in the next Proposition.

Proposition V.6. ([10]) Any improper $p \times m$ TFM $L(\lambda)$ has a realization

$$L(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B - \lambda F \\ \hline C & D \end{array} \right] \stackrel{\text{def}}{=} D + C(\lambda E - A)^{-1}(B - \lambda F), \quad (37)$$

and for any fixed $\alpha, \beta \in \mathbb{R}$, not both zero, there exists a realization

$$L(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B(\alpha - \lambda\beta) \\ \hline C & D \end{array} \right] \stackrel{\text{def}}{=} D + C(\lambda E - A)^{-1}B(\alpha - \lambda\beta), \quad (38)$$

where $A, E \in \mathbb{R}^{n \times n}$, $B, F \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and the pole pencil $A - \lambda E$ is regular. A realization (38) will be called centered at $\frac{\alpha}{\beta}$ (if $\beta = 0$ we interpret $\frac{\alpha}{\beta}$ as ∞). Occasionally, we shall use also the more compact notation $L(\lambda) = (A - \lambda E, B - \lambda F, C, D)$ and $L(\lambda) = (A - \lambda E, B(\alpha - \lambda\beta), C, D)$ to denote (37) and (38), respectively. Realizations of type (38) have been dubbed pencil realizations.

Definition V.7. ([10]) We call realizations of the type (37) or (38) minimal if the dimension of the square matrices A and E (also called the order of the realization) is as small as possible among all realizations of the respective kind.

Proposition V.8. ([10]) Any TFM $L(\lambda)$ has a minimal realization of type (37) of order equal to $\delta(L)$. For any fixed α and β , not both zero, and such that $\frac{\alpha}{\beta}$ is not a pole of $L(\lambda)$ there also exists a minimal realization of type (38) of order equal to $\delta(L)$. The condition imposed on $\frac{\alpha}{\beta}$ is needed only for writing down minimal realizations (38) which have order equal to $\delta(L)$. More precisely, even if $\frac{\alpha}{\beta}$ is a pole of $L(\lambda)$ we can still write a realization (38) but the minimal order will with necessity be greater than $\delta(L)$. This is exactly what is happening for realizations (36) which are obtained from (38) for $\alpha = 1$ and $\beta = 0$, and for which the minimal order is necessary greater than $\delta(L)$, provided $\frac{\alpha}{\beta} = \infty$ is a pole of $L(\lambda)$. Notice that for (38) we can always choose freely α and β such as to ensure $\frac{\alpha}{\beta}$ is not a pole of $L(\lambda)$. For the rest of the paper, if not otherwise stated, we assume this choice implicitly. The nice feature of (37) and (38) that their minimal order equals the McMillan degree of $L(\lambda)$ recommends them for the kind of problems treated in this paper.

Proposition V.9. ([10]) A given realization of type (37) of a TFM $L(\lambda)$ is minimal if and only if all of the following conditions hold true

$$\text{rank} \begin{bmatrix} A - \lambda E & B - \lambda F \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C}, \quad (39a)$$

$$\text{rank} \begin{bmatrix} E & F \end{bmatrix} = n, \quad (39b)$$

$$\text{rank} \begin{bmatrix} A - \lambda E \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C}, \quad (39c)$$

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n, \quad (39d)$$

while for realizations of type (38) similar conditions result by simply replacing (a) and (b) in (39) with

$$\text{rank} \begin{bmatrix} A - \lambda E & B(\alpha - \lambda\beta) \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C}, \quad (40a)$$

$$\text{rank} \begin{bmatrix} E & B \end{bmatrix} = n. \quad (40b)$$

Proposition V.10. Any two minimal realizations $L(\lambda) = (A - \lambda E, B(\alpha - \lambda\beta), C, D)$ and $L(\lambda) = (\tilde{A} - \lambda\tilde{E}, \tilde{B}(\alpha - \lambda\beta), \tilde{C}, \tilde{D})$ are always related by an equivalence transformation as

$$\tilde{E} = QEZ, \quad \tilde{A} = QAZ, \quad \tilde{B} = QB, \quad \tilde{C} = CZ, \quad \tilde{D} = D, \quad (41)$$

where Q and Z are unique invertible matrices.

APPENDIX B

Proof of Theorem III.6 We prove that any pair $(W(\lambda), V(\lambda))$ given by (19),(20) satisfies (18). We start with the equations (8)

$$\begin{bmatrix} \lambda I_p - A_{11} & -A_{12} \\ -A_{21} & \lambda I_{n-p} - A_{22} \end{bmatrix} \begin{bmatrix} Y(\lambda) \\ Z(\lambda) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(\lambda) \quad (42a)$$

$$Y(\lambda) = \begin{bmatrix} I_p & O \end{bmatrix} \begin{bmatrix} Y(\lambda) \\ Z(\lambda) \end{bmatrix} \quad (42b)$$

and apply a type (6) state equivalence transformation with

$$T = \begin{bmatrix} I_p & O \\ K & I_{n-p} \end{bmatrix} \quad (43)$$

where K can be any matrix in $\mathbb{R}^{(n-p) \times p}$, in order to get

$$\begin{bmatrix} \lambda I - (A_{11} - A_{12}K) & -A_{12} \\ (-KA_{11} - A_{21} + A_{22}K + KA_{12}K) & \lambda I - (A_{22} + KA_{12}) \end{bmatrix} \begin{bmatrix} Y(\lambda) \\ KY(\lambda) + Z(\lambda) \end{bmatrix} = \begin{bmatrix} B_1 \\ KB_1 + B_2 \end{bmatrix} U(\lambda) \quad (44a)$$

$$Y(\lambda) = \begin{bmatrix} I_p & O \end{bmatrix} \begin{bmatrix} Y(\lambda) \\ KY(\lambda) + Z(\lambda) \end{bmatrix} \quad (44b)$$

respectively. In a similar manner with getting from (8) to (10) via (9), we multiply (44a) to the left with the following invertible factor

$$\Omega_K(\lambda) = \begin{bmatrix} I_p & A_{12} \left(\lambda I_{n-p} - (A_{22} + K A_{12}) \right)^{-1} \\ O & I_{n-p} \end{bmatrix} \quad (45)$$

After the multiplication is performed, the first block row of the resulting equation yields $(\lambda I_p - W(\lambda))Y(\lambda) = V(\lambda)U(\lambda)$ which is exactly (18) with $W(\lambda)$ and $V(\lambda)$ having the expressions in (19) and (20), respectively. Finally, from the expression of $W(\lambda)$ in (19), clearly the McMillan degree of $W(\lambda)$ cannot exceed $(n - p)$.

Proof of Theorem IV.2 An equivalent condition for the pair $(\lambda I_p - W(\lambda), V(\lambda))$ to be coprime (over the compactification of \mathbb{C}) is for the compound transfer function matrix

$$\begin{bmatrix} (\lambda I_p - W(\lambda)) & V(\lambda) \end{bmatrix} \quad (46)$$

to have no (finite or infinite) Smith zeros (see [3], [9], [10] for equivalent characterizations of left coprimeness). According to [10, Theorem 2.1] (see also [3], [5]) the Smith zeros of (46) are among the Smith zeros (generalized eigenvalues) of the system–pencil of any minimal realization of (46). Hence we break this proof in two distinct parts: in part **I** we compute a type (38) *pencil realization* for (46) and prove that is indeed minimal, in the sense of Definition V.9. In part **II** of the proof we show that the system-pencil of the minimal realization from part **I** has no finite or infinite Smith zeros (generalized eigenvalues).

I We will show that the following type (38) *pencil realization* for $\begin{bmatrix} (\lambda I - W(\lambda)) & V(\lambda) \end{bmatrix}$ is a *minimal* realization in the sense of Definition V.9:

$$\begin{bmatrix} (\lambda I - W(\lambda)) & V(\lambda) \end{bmatrix} = \left[\begin{array}{cc|cc} (A_{22} + K A_{12}) - \lambda I_{n-p} & O & (A_{22}K + K A_{12}K - K A_{11} - A_{21}) & K B_1 + B_2 \\ O & I_p & I_p(\lambda_o - \lambda) & O \\ \hline A_{12} & I_p & \lambda_o I_p - A_{11} + A_{12}K & B_1 \end{array} \right] \quad (47)$$

I a) Observability for any finite $\lambda \in \mathbb{C}$ We note that

$$\begin{bmatrix} (A_{22} + K A_{12}) - \lambda I_{n-p} & O \\ O & I_p \\ A_{12} & I_p \end{bmatrix} = \begin{bmatrix} I & O & K \\ O & I & -K \\ O & O & I \end{bmatrix} \begin{bmatrix} A_{22} - \lambda I_{n-p} & O \\ O & I_p \\ A_{12} & I_p \end{bmatrix}$$

where the right hand side has full column rank for any $\lambda \in \mathbb{C}$, due to the observability of the pair (A_{12}, A_{22}) (from Assumption II.3). Hence point (39c) of Definition V.9 holds via the Popov–Belevitch–Hautus (PBH) criterion.

I b) Observability at $\lambda = \infty$ is equivalent via point (39d) of Definition V.9 with the following matrix having full column rank

$$\begin{bmatrix} I & O \\ O & O \\ A_{12} & I \end{bmatrix}.$$

I c) Controllability for any finite $\lambda \in \mathbb{C}$ We look at the following succession of equivalent singular matrix pencils

$$\begin{aligned} & \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & O & (A_{22}K + KA_{12}K - KA_{11} - A_{21}) & KB_1 + B_2 \\ O & I_p & I_p(\lambda_o - \lambda) & O \end{bmatrix} \sim \\ & \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & O & (\lambda K - KA_{11} - A_{21}) & KB_1 + B_2 \\ O & I_p & I_p(\lambda_o - \lambda) & O \end{bmatrix} \sim \\ & \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & K & (\lambda_o K - KA_{11} - A_{21}) & KB_1 + B_2 \\ O & I_p & I_p(\lambda_o - \lambda) & O \end{bmatrix} \sim \\ & \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & K & (-KA_{11} - A_{21}) & KB_1 + B_2 \\ O & I_p & -\lambda I_p & O \end{bmatrix} \sim \\ & \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & K & -A_{21} & KB_1 + B_2 \\ O & I_p & -\lambda I_p + A_{11} & O \end{bmatrix} \sim \\ & \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & K & -A_{21} & B_2 \\ O & I_p & -\lambda I_p + A_{11} & -B_1 \end{bmatrix} \sim \\ & \begin{bmatrix} A_{22} - \lambda I_{n-p} & K & -A_{21} & B_2 \\ -A_{12} & I_p & -\lambda I_p + A_{11} & -B_1 \end{bmatrix} \sim \begin{bmatrix} \lambda I_p - A_{11} & -A_{12} & B_1 & -I_p \\ -A_{21} & \lambda I_{n-p} - A_{22} & B_2 & K \end{bmatrix} \end{aligned} \quad (48)$$

The full row rank of the last pencil above for any $\lambda \in \mathbb{C}$, follows from the controlability Assumption IV.1 and the PBH criterion and it fulfills point (40a) of Definition V.9.

I d) Controlability at $\lambda = \infty$: is equivalent via point (40b) of Definition V.9 with the following matrix having full row rank

$$\begin{bmatrix} I_p & O & O & O \\ O & O & I_{n-p} & O \end{bmatrix}.$$

II) We look at the system-pencil of the realization (47), namely

$$\mathcal{S}(\lambda) \stackrel{def}{=} \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & O & (A_{22}K + KA_{12}K - KA_{11} - A_{21}) & KB_1 + B_2 \\ O & I_p & I_p(\lambda_o - \lambda) & O \\ A_{12} & I_p & \lambda_o I_p - A_{11} + A_{12}K & B_1 \end{bmatrix} \quad (49)$$

We will show next that the singular pencil in (49) has no finite or infinite Smith zeros (generalized eigenvalues), which will conclude that the pair $(\lambda I - W(\lambda), V(\lambda))$ is left coprime. We will show this, by proving that $\mathcal{S}(\lambda)$ keeps full row rank for any $\lambda \in \mathbb{C}$ and also for $\lambda = \infty$.

II a) No Finite Smith Zeros We look at the following succession of equivalent matrix pencils

$$\begin{aligned} & \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & O & (A_{22}K + KA_{12}K - KA_{11} - A_{21}) & KB_1 + B_2 \\ O & I_p & I_p(\lambda_o - \lambda) & O \\ A_{12} & I_p & \lambda_o I_p - A_{11} + A_{12}K & B_1 \end{bmatrix} \sim \\ & \begin{bmatrix} (A_{22} + KA_{12}) - \lambda I_{n-p} & O & (\lambda K - KA_{11} - A_{21}) & KB_1 + B_2 \\ O & I_p & I_p(\lambda_o - \lambda) & O \\ A_{12} & I_p & \lambda_o I_p - A_{11} & B_1 \end{bmatrix} \sim \\ & \begin{bmatrix} A_{22} - \lambda I_{n-p} & O & -A_{21} & B_2 \\ O & I_p & I_p(\lambda_o - \lambda) & O \\ A_{12} & I_p & \lambda_o I_p - A_{11} & B_1 \end{bmatrix} \sim \begin{bmatrix} A_{22} - \lambda I_{n-p} & O & -A_{21} & B_2 \\ A_{12} & I_p & \lambda_o I_p - A_{11} & B_1 \\ O & I_p & I_p(\lambda_o - \lambda) & O \end{bmatrix} \sim \\ & \begin{bmatrix} A_{22} - \lambda I_{n-p} & O & -A_{21} & B_2 \\ A_{12} & O & \lambda I_p - A_{11} & B_1 \\ O & I_p & I_p(\lambda_o - \lambda) & O \end{bmatrix} \sim \begin{bmatrix} A_{11} - \lambda I_p & A_{12} & B_1 & O \\ A_{21} & A_{22} - \lambda I_{n-p} & B_2 & O \\ \lambda I_p & O & O & I_p \end{bmatrix} \end{aligned}$$

The last pencil above clearly holds full row rank for any $\lambda \in \mathbb{C}$ due to Assumption IV.1 and the PBH criterion.

II b) No Smith Zeros at Infinity: Follows by the adaptation of [5, Lemma 1].

Proof of Lemma IV.4 This proof is based entirely on [10, Theorem 3.1] (Basic Pole Displacement Result). We start with the following type (37) *minimal* realization of

$$\left[\begin{array}{cc|cc} I_p & O & I_p(\lambda_o - \lambda) & O \\ (\lambda I_p - W(\lambda)) & V(\lambda) & (A_{22}K + KA_{12}K - KA_{11} - A_{21}) & KB_1 + B_2 \\ \hline I_p & A_{12} & \lambda_o I_p - A_{11} + A_{12}K & B_1 \end{array} \right] \quad (50)$$

It can be observed that (50) is already in the ordered block-Schur form [10, (2.14)/pp. 252]. We want to employ [10, Theorem 3.1] in order to compute the invertible TFM from [10, (3.1)/pp. 252] which we denote with $\Theta(\lambda)$ that by premultiplying (50) will cancel out the p poles at infinity of (50). Any type (38) realization of a valid $\Theta(\lambda)$ satisfies [10, (3.2)/pp. 252] for certain *invertible* X and Y matrices. Hence for any

$$\Theta(\lambda) = \left[\begin{array}{c|c} A_x - \lambda I_p & B_x(\lambda - \lambda_o) \\ \hline C_x & D_x \end{array} \right]$$

(with D_x must be invertible because $\Theta(\lambda)$ is invertible) we write the conditions from [10, (3.2)/pp. 252] which are equivalent with

$$\left[\begin{array}{cc} A_x - \lambda I_p & B_x(\lambda - \lambda_o) \\ C_x & D_x \end{array} \right] \left[\begin{array}{c} X \\ I \end{array} \right] = \left[\begin{array}{c} Y \\ O \end{array} \right] (I_p - \lambda O) \quad (51)$$

From the first block row of (51) we get that $C_x X = -D_x$ and from the second block-row of (51) we get $B_x(\lambda - \lambda_o) = Y - (A_x - \lambda I_p)X$. Consequently

$$\Theta(\lambda) = \left[\begin{array}{c|c} A_x - \lambda I_p & Y - (A_x - \lambda I_p)X \\ \hline C_x & -C_x X \end{array} \right]$$

which is equivalent with

$$\Theta(\lambda) = \left[\begin{array}{c|c} A_x - \lambda I_p & Y \\ \hline C_x & O \end{array} \right]$$

where C_x and Y are arbitrary invertible matrices. We have denoted C_x with T_4 and we have denote Y with T_5 to avoid notational confusion. The proof ends.